



1.6.13 Polynomials with real coefficients

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Outline

- In this topic, we will
 - Examine the properties of the roots of polynomials with real coefficients
 - Look at some examples
 - Emphasize that this is one of the most important results in this course





Polynomials with real coefficients

- A polynomial with real coefficients is any polynomial where all coefficients are real.

- For example ,

$$z^5 + 4.2z^4 + 2.7z^3 + 9.3z^2 - 0.8z + 1.2$$

- Any such polynomial of degree n may be written as

$$\begin{aligned} p(z) &= \sum_{k=0}^n \alpha_k z^k \\ &= \alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_2 z^2 + \alpha_1 z + \alpha_0 \end{aligned}$$

- Important: the coefficient of z^k is α_k





Properties of the complex conjugate

- Let us review the properties of the complex conjugate:
 - For any complex z : $(z^*)^* = z$
 - For a real number α , $\alpha^* = \alpha$
 - For the sum of two complex numbers, $(z_1 + z_2)^* = z_1^* + z_2^*$
 - Consequently, $\left(\sum_{k=1}^n z_k\right)^* = \sum_{k=1}^n z_k^*$
 - For the product of two complex numbers, $(z_1 z_2)^* = z_1^* z_2^*$
 - For a complex number raised to an integer exponent, $(z^n)^* = (z^*)^n$





Polynomials with real coefficients

- Suppose that r is a root of a complex polynomial with real coefficients:

$$p(r) = \sum_{k=0}^n \alpha_k r^k = 0$$

- We have two possibilities: either r is real, or r is not real
 - Let us suppose that r is not real





Polynomials with real coefficients

- If r is not real, let us conjugate both sides of $p(r) = \sum_{k=0}^n \alpha_k r^k = 0$

$$\left(\sum_{k=0}^n \alpha_k r^k \right)^* = 0^*$$

$$\sum_{k=0}^n (\alpha_k r^k)^* = 0$$

$$\sum_{k=0}^n \alpha_k^* (r^k)^* = 0$$

But each $\alpha_k^* = \alpha_k$

$$\sum_{k=0}^n \alpha_k (r^*)^k = 0$$

- Is this not $p(r^*) = \sum_{k=0}^n \alpha_k (r^*)^k$?

– Consequently, if r is a root, the r^* is also a root





Polynomials with real coefficients

Theorem:

If p is a polynomial with real coefficients, and r is a root, then so is r^* .

Proof: Assume $p(r) = \sum_{k=0}^n \alpha_k r^k = 0$.

$$p(r^*) = \sum_{k=0}^n \alpha_k (r^*)^k$$

$$= \sum_{k=0}^n \alpha_k (r^k)^*$$

$$= \sum_{k=0}^n \alpha_k^* (r^k)^*$$

$$= \sum_{k=0}^n (\alpha_k r^k)^*$$

$$= \left(\sum_{k=0}^n \alpha_k r^k \right)^* = 0^* = 0 \quad \blacksquare$$

But each $\alpha_k^* = \alpha_k$





Polynomials with real coefficients

- To reiterate:
 - If p is a polynomial with real coefficients and r is a non-real complex root of p then r^* is also a root of p
 - We will describe r and r^* as a *complex-conjugate pair*
 - More generally, if p is a polynomial with real coefficients and r is a non-real complex root with multiplicity m , then r^* is also a root of p , also of multiplicity m





Polynomials with non-real coefficients

- Note that if the coefficients are not real, then the correct statement is

If r is a root of the polynomial p ,
then r^* is a root of the polynomial with the
conjugate of the coefficients

- For example, $r = 2 - j$ is a root of $z^2 + (1 - 4j)z - 1 + 13j$
 - Therefore, $r^* = 2 + j$ is a root of $z^2 + (1 + 4j)z - 1 - 13j$
 - Evaluating the first polynomial at $r^* = 2 + j$ results in $8 + 10j \neq 0$





Examples

- To demonstrate,
 - We already know the roots of $z^2 + 1$ are j and $-j$
 - The roots of $z^2 + bz + c$ are

$$-\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}$$

- If $b^2 - 4c \geq 0$, then both roots are real
- If $b^2 - 4c < 0$, then the roots are

$$-\frac{b}{2} \pm \frac{\sqrt{4c - b^2}}{2} j$$



Examples

- To demonstrate,
 - Consider the polynomial

$$z^5 + 3z^4 + 7z^3 + 5z^2 + 2z + 1$$

- Its roots are:

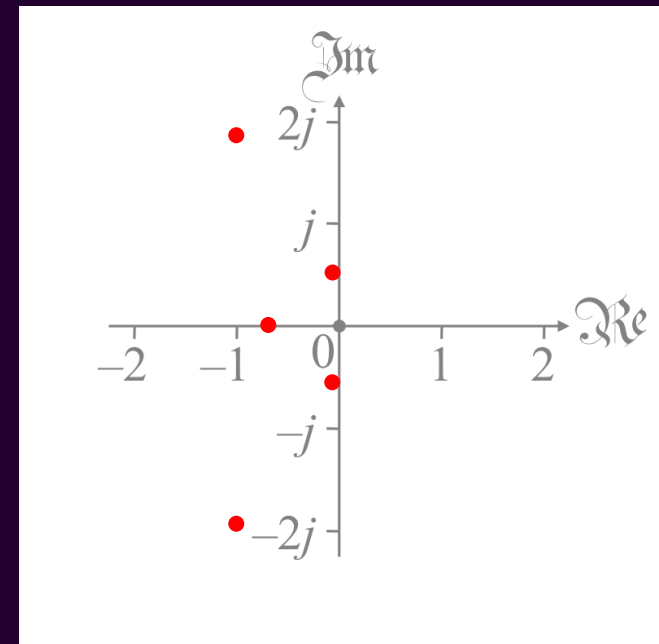
$$-0.035181403011327564 + 0.51437834108030281j$$

$$-0.035181403011327564 - 0.51437834108030281j$$

$$-1.0777404719073206 + 1.9229758521534597j$$

$$-1.0777404719073206 - 1.9229758521534597j$$

$$-0.77415625016270363$$





Examples

- To demonstrate,
 - Consider the polynomial

$$z^5 + 4z^4 + 7z^3 + 12z^2 + 10z + 2$$

- Its roots are:

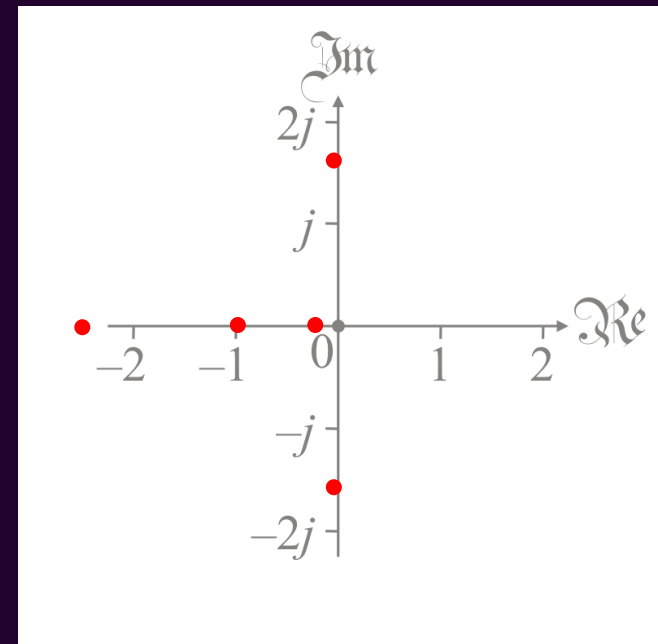
–0.28217940701111523

–1

–2.5426012611094618

–0.087609665939711494 + 1.6673028066114304j

–0.087609665939711494 – 1.6673028066114304j





Summary

- In this topic, you now
 - Understand the significance of the roots of a polynomial with real coefficients
 - Their roots are either real or come in complex conjugate pairs
 - Have been exposed to the proof
 - Are aware of some examples of this result
 - Should be aware that this will be one of the most significant results in this course



References

- [1] https://en.wikipedia.org/wiki/Complex_conjugate_root_theorem



Acknowledgments

None so far.



Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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